The Weekly Rigor

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Verifying Trigonometric Identities with Simple Arguments Involving the Product of Three Trigonometric Functions: Problems with Solutions

(Part 1)

INTRODUCTION

This set of problems depend on the following trigonometric identities as given:

$\sec\theta = \frac{1}{\cos\theta}$. $\csc\theta = \frac{1}{\sin\theta}$. $\cot\theta = \frac{1}{\tan\theta}$. (Reciprocal Identities)

$\tan\theta = \frac{\sin\theta}{\cos\theta}$. $\cot\theta = \frac{\cos\theta}{\sin\theta}$. (Ratio Identities)

$$\sin^2\theta + \cos^2\theta = 1.$$

$$\tan^2\theta + 1 = \sec^2\theta.$$ (Pythagorean Identities)

$$1 + \cot^2\theta = \csc^2\theta.$$

Strictly speaking, the identity $\cot\theta = \frac{\cos\theta}{\sin\theta}$ should be regarded as a corollary of some of the other identities, viz.,

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{\left(\frac{\sin\theta}{\cos\theta}\right)} = \frac{1}{1}\cdot\frac{\cos\theta}{\sin\theta} = \frac{\cos\theta}{\sin\theta}.$$

Similarly, the second and third Pythagorean Identities follow from the first. To wit:

$$\tan^2\theta + 1 = \frac{\sin^2\theta}{\cos^2\theta} + 1 = \frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} = \sec^2\theta, \text{ and}$$

$$1 + \cot^2\theta = 1 + \frac{\cos^2\theta}{\sin^2\theta} = \frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} = \csc^2\theta.$$

Additional useful corollaries are the following:

$$\cos\theta = \frac{1}{\sec\theta}. \qquad \sin\theta = \frac{1}{\csc\theta}. \qquad \tan\theta = \frac{1}{\cot\theta}.$$

Proofs: $$\frac{1}{\sec\theta} = \frac{1}{\left(\frac{1}{\cos\theta}\right)} = \frac{1}{1}\cdot\frac{\cos\theta}{1} = \cos\theta.$$

$$\frac{1}{\csc\theta} = \frac{1}{\left(\frac{1}{\sin\theta}\right)} = \frac{1}{1}\cdot\frac{\sin\theta}{1} = \sin\theta.$$

$$\frac{1}{\cot\theta} = \frac{1}{\left(\frac{1}{\tan\theta}\right)} = \frac{1}{1}\cdot\frac{\tan\theta}{1} = \tan\theta.$$

A last set of such useful corollaries are the following:

$$\sin^2\theta = 1 - \cos^2\theta. \qquad \cos^2\theta = 1 - \sin^2\theta. \qquad \tan^2\theta = \sec^2\theta - 1.$$

$$1 = \sec^2\theta - \tan^2\theta. \qquad 1 = \csc^2\theta - \cot^2\theta. \qquad \cot^2\theta = \csc^2\theta - 1.$$

Proofs: $$1 - \cos^2\theta = \left(\sin^2\theta + \cos^2\theta\right) - \cos^2\theta = \sin^2\theta.$$

$$1 - \sin^2\theta = \left(\sin^2\theta + \cos^2\theta\right) - \sin^2\theta = \cos^2\theta.$$

$$\sec^2\theta - 1 = \left(\tan^2\theta + 1\right) - 1 = \tan^2\theta.$$

$$\sec^2\theta - \tan^2\theta = \left(\tan^2\theta + 1\right) - \tan^2\theta = 1.$$

$$\csc^2\theta - \cot^2\theta = \left(1 + \cot^2\theta\right) - \cot^2\theta = 1.$$

$$\csc^2\theta - 1 = \left(1 + \cot^2\theta\right) - 1 = \cot^2\theta.$$